Analysis of resonances in action space for symplectic maps

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The network of resonances in the action plane for a four-dimensional map is obtained by computing the actions from the Fourier coefficients of the orbits, and it is compared with the results of Birkhoff normal forms. This method, which combines the positive features of standard frequency analysis and normal forms, is suitable to study the one turn map of a particle accelerator. [S1063-651X(97)12312-1]

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The phase space of four dimensional symplectic maps is difficult to explore. These maps describe the motion of a charge in the transverse section of a particle accelerator, or more generally the Poincaré section of a Hamiltonian system with three degrees of freedom. The presence of an elliptic fixed point, corresponding to a close stable orbit along the lattice, allows us to use Birkhoff normal forms, and to introduce a Hamiltonian H, which interpolates the orbits and defines approximate invariants [1,2]. Close to the origin the normal forms reproduce fairly well the numerical iterations of the map (tracking) [3,4], but the agreement becomes poor when the short term stability boundary is approached. The frequency analysis, first introduced in celestial mechanics [5,6], has been used to describe the whole stability region [7,8]. Strategies to compute the invariants by solving the Hamilton-Jacobi equation have been previously considered [9]. We propose to compute the Fourier coefficients and the invariant actions for every orbit with nonresonant frequencies. The decay of the Fourier amplitudes determines the analyticity strip of the KAM tori, whereas the map between the nonresonant frequencies and the actions provides an invariant picture of the nonlinear resonances. The image in the action space of a neighborhood of the origin is a set with empty channels corresponding to the resonances (since the actions are not defined on resonant orbits). The resonance associated with a given channel is identified either by nonresonant normal forms or by using the frequency map.

Given any orbit $[x_1(n),p_1(n),x_2(n),p_2(n)]$ of a symplectic map, we consider the following representation as a Fourier series:

$$x_1(n)-ip_1(n)=\sum_{k_1,k_2} e^{i(\Omega_1 k_1+\Omega_2 k_2)n} a_{k_1,k_2}, \tag{1}$$

$$x_2(n)-ip_2(n)=\sum_{k_1,k_2} e^{i(\Omega_1 k_1+\Omega_2 k_2)n} b_{k_1,k_2}.$$

By using a Hanning filter [5,10], a high accuracy is achieved for the frequencies Ω_1 and Ω_2, with a small sample of data (the error decreases asymptotically as N^{-4} [10]). If the frequencies are nonresonant, the orbit is dense on a two dimensional (2D) torus, whose parametrization is given by Eq. (1), where $\Omega_1 n$ and $\Omega_2 n$ are replaced by the continuous variables Θ_1 and Θ_2. Among the coefficients a_{k_1,k_2} and b_{k_1,k_2}, which asymptotically have an exponential decay with $|k_1|+|k_2|$, only the K leading terms are kept. The accuracy of the interpolation, depending on K and the length N of the orbit, is measured by the mean square deviation σ between the true orbit and its Fourier reconstruction. By choosing N large enough for a fixed K, the desired accuracy for the frequencies and the K leading coefficients can be achieved. In order to keep σ constant, we have to increase K while moving away from the origin, since the tori are more perturbed and their analyticity strip is thinner. Conversely, keeping K constant, we expect σ to grow with the distance from the origin.

Given an orbit with nonresonant frequencies, we consider the torus parametrized by the angles Θ_1 and Θ_2. By fixing Θ_2 or Θ_1 we obtain the basic cycles γ_1 or γ_2, namely, the 1D tori whose topological product is the 2D torus itself. The invariant actions are defined by

$$J_i=\frac{1}{2\pi}\oint_{\gamma_i}(p_1 dx_1+p_2 dx_2)$$
$$=\frac{1}{2\pi}\int_0^{2\pi}\left(p_1\frac{\partial x_1}{\partial\Theta_i}+p_2\frac{\partial x_2}{\partial\Theta_i}\right)d\Theta_i, \quad i=1,2. \tag{2}$$

Since the action J_1 (J_2) is independent of the choice of the angle Θ_2 (Θ_1), after averaging on it we have

$$J_i=\tfrac{1}{2}\sum_{k_1,k_2} k_i(|a_{k_1k_2}|^2+|b_{k_1k_2}|^2), \quad i=1,2. \tag{3}$$

If the nonlinear frequencies Ω_1 and Ω_2 satisfy resonant conditions

$$q_1\Omega_1+q_2\Omega_2=2\pi\ell, \quad q_1,q_2,\ell\in N, \tag{4}$$

the actions are no longer defined. For a map with linear frequencies ω_1 and ω_2, the Birkhoff normal form defines a Hamiltonian H which interpolates the orbits [1,2],

$$H=\sum_{k_1,k_2} H_{k_1,k_1}(J_1,J_2)\cos(k_1\theta_1+k_2\theta_2+\delta_{k_1,k_2}), \tag{5}$$

where H_{k_1,k_2} is a polynomial in $J_1^{1/2}$ and $J_2^{1/2}$. If the linear frequencies are nonresonant only the $k_1=k_2=0$ terms con-

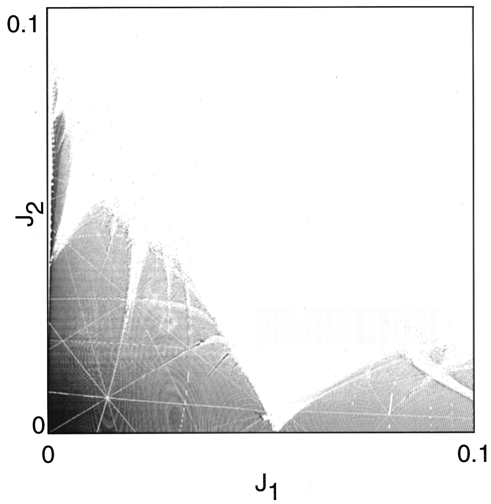

FIG. 1. Plot of the resonance lines on the actions plane for the 4D Hénon map with linear frequencies $(0.28\times 2\pi, 0.31\times 2\pi)$.

tribute to H, and the actions are invariant. The Hamiltonian H, truncated at order M, explicitly reads

$$H = H_{0,0}(J_1, J_2) = \omega_1 J_1 + \omega_2 J_2 + \sum_{m\geq 2}^{M} \sum_{m_1+m_2=m} h_{m_1,m_2} J_1^{m_1} J_2^{m_2} \qquad (6)$$

(we denote the invariant actions by capital letters). The resonance condition (4) on the frequencies $\Omega_i = \partial H/\partial J_i$ reads

$$q_1\omega_1 + q_2\omega_2 - 2\pi\ell + \sum_{m\geq 1}^{M-1} \sum_{m_1+m_2=m} [q_1(m_1+1)h_{m_1+1,m_2} + q_2(m_2+1)h_{m_1,m_2+1}] J_1^{m_1} J_2^{m_2} = 0. \qquad (7)$$

At the lowest truncation order $M=2$, the curves defined by Eq. (7) are straight lines, and at the next order $M=3$ they are arcs of conics.

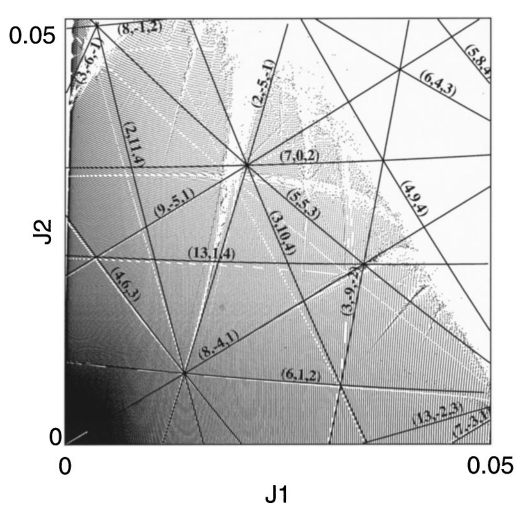

FIG. 2. Comparison between the resonance lines computed numerically and using the nonresonant normal from at order 4, for the 4D Hénon map with linear frequencies $(0.28\times 2\pi, 0.31\times 2\pi)$.

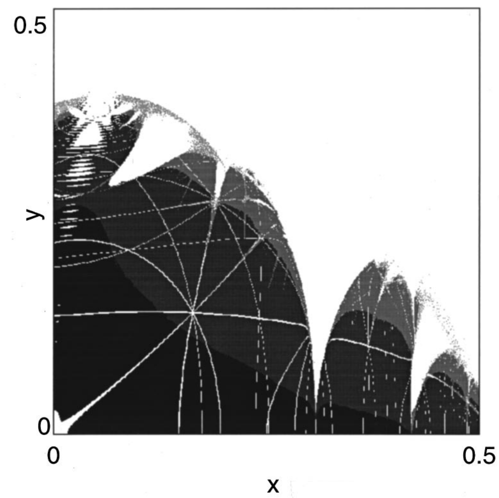

FIG. 3. Mean square deviation of the signal from its Fourier reconstruction of the 4D Hénon map with linear frequencies $(0.28\times 2\pi, 0.31\times 2\pi)$. The initial points are taken in the $p_x=0$, $p_y=0$ plane, and are plotted in the (x,y) plane with four tones of gray of increasing darkness corresponding to $2\times 10^{-2} > \sigma \geq 10^{-2}$, $10^{-2} > \sigma \geq 5\times 10^{-3}$ $5\times 10^{-3} > \sigma \geq 10^{-3}$, and $\sigma_z \leq 10^{-3}$.

The following strategy is used to compute the resonance lines from tracking. All the points of a regular 2D lattice in a section plane of phase space are iterated N times, and the corresponding orbits are Fourier analyzed. After discarding the points whose orbits have resonant frequencies, the actions are computed by Eq. (3) for the remaining ones: the plot of these points in the (J_1, J_2) plane exhibits empty channels corresponding to the resonances. In Fig. 1 we show, for the quadratic Hénon map with linear frequencies $\omega_1 = 0.28\times 2\pi$ and $\omega_2 = 0.31\times 2\pi$, the results obtained by choosing 600×600 points in the square $0\leq x\leq 0.5, 0\leq y\leq 0.5$, and $p_x = p_y = 0$. The integer vector corresponding to any resonance (q_1, q_2, ℓ) can be read in Fig. 2, where the resonance

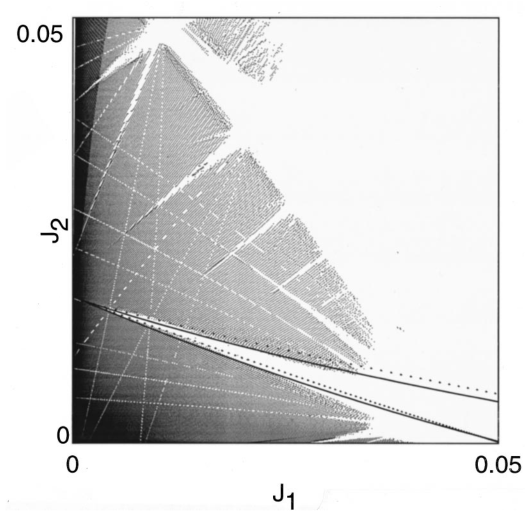

FIG. 4. Comparison between the amplitude of the (5,0) resonance of the Hénon map with linear frequencies $(0.205\times 2\pi, 0.618034\times 2\pi)$ computed numerically and with a quasiresonant normal form. The truncation order of H in Cartesian coordinates are 8 (continuous line) and 6 (dashed line).

lines, computed from Eq. (7) with $M=2$, are shown. The agreement is excellent near the origin, where the resonance width is below the pixel dimension.

We computed the mean square deviation σ of the Fourier interpolation with respect to the orbit for fixed values of K and N. To every initial condition in the square $0 \leqslant x \leqslant 0.5$, $0 \leqslant y \leqslant 0.5$, and $p_x = p_y = 0$, a tone of gray has been assigned according to the value of σ, having chosen $N = 10^4$ and $K=10$. In Fig. 3, we show the result of this analysis for the Hénon map with linear frequencies $\omega_1 = 0.205 \times 2\pi$ and $\omega_2 = (\sqrt{5}-1)\times\pi$ close to the resonance (5,0,1). The trajectories with $\sigma > 2\times10^{-2}$ have been dropped from the figure, since we assume them to be chaotic, or at the border of a chaotic region. This conjecture has been numerically checked for some orbits. We compare these results with the quasiresonant normal form for the (5,0,1) resonance, whose lowest order Hamiltonian reads $H = h_{0,0}(j_1, j_2) + A j_1^{5/2} \cos(5\theta_1 + \delta)$, where $h_{0,0}$ and $A j_1^{5/2}$ are the leading terms of $H_{0,0}$ and $H_{5,0}$. Above and below the separatrix, the invariant action J_1 is given by $J_1 = (2\pi)^{-1}\int_0^{2\pi} j_1(\theta_1, j_2; E)\, d\theta_1$, where j_1 is defined implicitly by $H(j_1, j_2, \theta_1) = E$, and E is fixed by the initial condition [11]. Correspondingly the angle Θ_1 is computed and, together with $J_2 = j_2$, $\Theta_2 = \theta_2$ defines the canonical transformation to the new variables. By computing the solutions $J_{1\pm}$ corresponding to the separatrix, where E is the value H takes at the hyperbolic points, we evaluate $J_{1\pm}(J_2)$, which gives the solid and dotted lines of Fig. 4 corresponding to truncation order $M=4$ and 3 (the truncation order of H is $2M$ referred to cartesian coordinates).

The resonance plot in action space provides an effective picture of the resonance structures, and is suitable to investigate the chaotic regions near the short term stability border. The invariance of actions for a small periodic modulation of the linear frequencies, and consequently the stability of the resonance patterns in action space, have been checked. The computation of the resonance lines with the lowest order normal forms is straightforward, whereas the analytic determination of the resonance width is more tricky, since it requires the computation of the corresponding quasiresonant normal form and the numerical evaluation of the invariants. The proposed method is suitable to investigate the betatronic motion of realistic accelerator models or the data taken from experiments with pencil beams, since it provides a picture of nonlinear resonances in the space of invariants, with the same computational load as the standard frequency analysis.

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